First-order ordinary differential equations: linear equations

We say that an ODE is **LINEAR** if its general representation is:

$$y' + P(x) \cdot y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions of x.

In particular, if $Q(x) = 0$, then the equation can be solved by **SEPARABLE VARIABLES**. That is,

$$y' + P(x) \cdot y = 0$$

$$\Rightarrow \frac{dy}{dx} + P(x) \cdot y = 0 \Rightarrow \int \frac{1}{y}\,dy = -\int P(x)\,dx \Rightarrow y + C = e^{-\int P(x)\,dx}$$

If, as is typically the case, $Q(x) \neq 0$, then we must apply a resolution method based on a convenient substitution by Lagrange. We define:

$$y = u(x) \cdot v(x)$$

Thus, $y' = u' \cdot v + u \cdot v'$ (applying the product rule).

Replacing this into the original ODE:

$$y' + P(x) \cdot y = Q(x)$$

$$(u' \cdot v + u \cdot v') + P(x) \cdot (u \cdot v) = Q(x)$$

Rearranging, we get:

$$(u' + P(x) \cdot u) \cdot v + u \cdot v' = Q(x)$$

Then, a system of two simpler equations can be formed as follows:

$$\begin{cases} u' + P(x) \cdot u = 0 \\ u \cdot v' = Q(x) \end{cases}$$

Where **the first equation is solved by Separable Variables**.

From the second equation, we solve for v by substituting the value of $u(x)$ obtained in the first equation.

Solving the System

$$\begin{cases} u' + P(x) \cdot u = 0 \\ u \cdot v' = Q(x) \end{cases}$$

Where **the first equation is solved by Separable Variables**.

$$u' + P(x) \cdot u = 0 \quad \Rightarrow \quad \frac{du}{dx} = -P(x)u \quad \Rightarrow \quad \int \frac{1}{u}\,du = -\int P(x)\,dx$$

$$\therefore \quad u(x) = e^{-\int P(x)\,dx}$$

And from the second, we solve for v by substituting the value of $u(x)$ obtained from the first equation.

$$u \cdot v' = Q(x) \quad \Rightarrow \quad e^{-\int P(x)\,dx} \cdot v' = Q(x) \quad \Rightarrow \quad v' = Q(x)e^{\int P(x)\,dx}$$

$$\therefore \quad y(x) = u(x) \cdot v(x) = e^{-\int P(x)\,dx} \left(\int Q(x)e^{\int P(x)\,dx}\,dx + C \right)$$

Example

$$y' + \sin(x)\, y = 2x\, e^{\cos(x)}$$

Where $P(x) = \sin(x)$; $Q(x) = 2x\, e^{\cos(x)}$.
Continuing with Lagrange's convenient substitution. We let: $y = u(x) \cdot v(x)$.
Therefore, $y' = u'\, v + u\, v'$. Then, substituting into the original ODE:

$$y' + \sin(x) \cdot y = 2x\, e^{\cos(x)}$$

$$(u' \cdot v + u \cdot v') + \sin(x) \cdot (u \cdot v) = 2x\, e^{\cos(x)}$$

Regrouping, we have:

$$(u' + \sin(x) \cdot u) \cdot v + u \cdot v' = 2x\, e^{\cos(x)}$$

$$\Rightarrow \begin{cases} u' + \sin(x) \cdot u = 0 \\ u \cdot v' = 2x\, e^{\cos(x)} \end{cases}$$

Where **the first equation is solved by Separable Variables**.
And from the second, we solve for v by substituting the value of $u(x)$ obtained from the first equation.
Solving the system:

$$\begin{cases} u' + \sin(x) \cdot u = 0 \\ u \cdot v' = 2x\, e^{\cos(x)} \end{cases}$$

Where **the first equation is solved by Separable Variables**.

$$u' + \sin(x) \cdot u = 0 \quad \Rightarrow \quad \frac{du}{dx} = -\sin(x)u \quad \Rightarrow \quad \int \frac{1}{u}\, du = -\int \sin(x)\, dx$$

$$\therefore \quad u(x) = e^{\cos(x)}$$

And from the second, **we solve for v by substituting the value of $u(x)$ obtained from the first equation**.

$$u \cdot v' = 2x\, e^{\cos(x)} \quad \Rightarrow \quad e^{\cos(x)} v' = 2x \quad \Rightarrow \quad v' = \frac{2x}{e^{\cos(x)}}$$

$$v(x) = x^2 + C$$

$$\therefore \quad y(x) = u(x) \cdot v(x) = e^{\cos(x)} \left(x^2 + C\right)$$